JOURNAL OF MATERIALS SCIENCE 15 (1980) 535–556

Tewksbury Lecture: Putting fracture to work

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It is emphasized that the science and crafts associated with the parting of solids are of considerable industrial, cultural and historical interest. Some principles of the modern theory of fracture which may be relevant to the controlled separation of a solid into pieces are reviewed. The use of path independent integrals in the analysis of indentation fracture is discussed, and some of the subtleties involved in treating the motion, deviation and forking of cracks and the energy balance in crushing and shattering are considered. The paper concludes with a brief account of some recent work on the theory of flint knapping and of the influence of the environment on the fracture process.

1. Introduction

It is only common prudence to consider the possible failure of anything built or constructed, so that when fracture is mentioned perhaps the first thought that springs to mind is how it may be prevented. Concern for safety and the protection of the environment have caused this aspect of the subject to grow in importance with the scale of engineering structures and the increasingly awesome consequences of their failure, particularly in the fields of nuclear power and the transport and production of chemicals and fuels. In this Fourth Tewksbury Symposium we look at the other side of the coin and consider the parting of solids as a useful art. Today this activity is the basis of great industries concerned with blasting, mining, tunnelling, quarrying and the drilling of rocks; with the machining, grinding and cutting of all materials; with some forms of dredging and metal working; with comminution, an essential starting point for many processes, including the winning of most metals, the burning of solid fuels, and the preparation of components from powders; and, regrettably, with many aspects of the hideous arts of war. We depend upon it for our paints, cosmetics and doctors' pills, as the pestle of the apothecary reminds us. From the parting of solids stem the arts of the sculptor and carver; the crafts of the mason and worker of gems. Among its tools are the razor, the scissors and the scalpel. We all know it, as master or servant, in the cutting of glass and tiles and the working of wood, and in many other homely tasks about the kitchen and garden. It influences the texture of our baking, and in common with those many animals which rely on beak, tooth, pincer or claw, enables us to win and eat our food. The ability to break or tear things is indeed a much older concern of mankind than keeping aircraft flying, or ensuring the safety of nuclear reactors or structures in the North Sea. Flint knapping and the development of sharp tools and weapons are elements in the very history of man, and the products of the mill and grindstone were his earliest manufactures.

These things have shaped our thought and speech. "Eyeless in Gaza, at the mill with slaves", our life is "one dem'd horrid grind". But "set your faces like a flint"; "in time small wedges cleave the hardest oak, in time the flint is pierced with softest shower". Though, "unkindest cut", "more water glideth by the mill than wot's the miller of", still "Gottesmühlen mahlen langsam, mahlen aber trefflich klein".

Breaking things and preventing their failure thus account for considerable human endeavour. The student of fracture is fortunate in having a subject of such widespread interest and importance. Not only are its principles relevant to such extremes as the earthquake and the microtome, but its understanding must embrace, besides the

This paper is the text of the Fourth Tewksbury Lecture on Fracture, delivered 12 February 1979, at the University of Melbourne, Australia.

0022–2461/80/030535–22$04.20/0

fracture itself, all the processes determining the deformation and strength of the material before and during this event, and the influence of the environment upon them.

To so broad a canvas we can add here only a fine detail or some general perspective. The symposium is to include lectures by experts in a number of fields we have mentioned. Accordingly we shall attempt in this introductory lecture only to discuss a few general principles which may be relevant to them. We begin with a brief recapitulation of some of the salient ideas in the current theory of fracture.

A solid may be separated into parts by three processes. Rows of atoms or molecules may be pulled apart normal to their centres of mass (*cracking*); these rows may slide over one another until they part company (*sliding off*); *individual atoms may be removed* as in vacancy migration or electrochemical attack. For any of these processes to occur, two conditions must be satisfied. First, the total energy of the body and loading system (if such a division is convenient) must fall as the fracture proceeds. Secondly, the physical mechanism causing the fracture must be able to operate – for example, a sufficient stress to cause separation or sliding must be present, or the temperature must be high enough for individual atoms to jump.

Although it is strictly necessary always to take account of thermal motion on an atomic scale and to make a kinetic model of cracking and sliding off [1–5], even at ordinary temperatures, it is sometimes a convenient idealization to ignore the thermal motion and to work with purely mechanical (though possibly rate dependent) theories. The realization that most engineering materials either contain small crack-like defects or readily develop them in service has caused considerable emphasis to be placed recently on the understanding and control of the propagation of cracks. Their initiation and behaviour on an atomic scale is nevertheless still a very important concern in the fundamental study of fracture processes, particularly those due to creep, fatigue and electrochemical attack. Study of the microscopic processes also plays an important part in the development of materials of greater toughness [6–8]. Indeed, these aspects of the subject currently need perhaps more emphasis than engineering fracture mechanics, for our knowledge of them is less complete. The principles of fracture mechanics however are universal, equally applicable to macroscopic cracks in engineering structures and to defects on an atomic scale. Thus the large development of the subject on the engineering side is at the same time reacting with advantage on studies of the micro-processes involved.

Another useful idealization in considering crack propagation is to distinguish fractures where the material remains essentially elastic except in regions near the crack tip whose linear dimensions are small compared with the crack length, and those where it does not. When the non-elastic region is confined to the neighbourhood of the crack tip in this sense we can decouple the energy available to drive the crack and that required to produce fracture, even though the latter may be much greater than the ideal fracture energy for the brittle separation of atomic planes [10]. Griffith [11, 12] was the first to recognize the principles governing crack propagation in this situation, but we now usually formulate them in terms of the energy release rate or crack extension force [9] G and write the necessary condition for crack advance as

$$G = R \tag{1}$$

where R is the fracture energy.

The classical situation in which the crack stability can be analysed in this way is the ideally brittle cleavage of a linear elastic material. Then $R = 2\gamma$ where γ is the surface energy. The singular stresses p_{ij} near the crack tip are given by

$$p_{ij} = (2\pi r)^{-1/2} K_s f_{sij}(\theta) \tag{2}$$

where r is the distance from the crack tip, and K_1, K_2, K_3 are the stress intensity factors for the loading modes I, II and III respectively. For mode I loading,

$$G = \frac{1-\nu}{2\mu} K_1^2 \tag{3}$$

where ν is Poisson's ratio and μ the shear modulus.

A much wider class of fractures can be discussed in this same general framework by including in R the energy absorbed by all the non-linear processes which contribute directly to the tearing apart which occurs when the new surfaces are formed [9, 10]. This was shown to great effect by Cottrell in the First Tewksbury Lecture [13], by using the DBCS model [14–20] to provide a theory of the fracture energy $R = \sigma_1 \phi_c$ depending on two parameters: σ_1, the strength of the layer ahead of the crack and ϕ_c, the critical crack tip displacement. The DBCS model has been very widely applied and developed, as discussed in recent reviews [5, 21–24]. The theory gives the fracture stress σ_f as [13, 17]

$$\sigma_f/\sigma_1 = (2/\pi)\cos^{-1}\{\exp(-c^*\pi/c)\} \qquad (4)$$

where $c^* = M\phi_c/4\sigma_1$ and M is an elastic modulus ($=\mu(1-\nu)$ for plane strain, $\mu(1+\nu)$ for plane stress and μ for anti-plane strain). The condition $c = c^*$ defines the crack length at which the material becomes *notch sensitive* [13]. When $c \gg c^*$, Equation 4 reduces to the Griffith condition and we have a dangerous "low-stress" failure of the cumulative [25] or localized [21] type with $\sigma_f \ll \sigma_1$. For $c < c^*$, σ_f approaches σ_1, the strength of the layer ahead of the crack and the material is not notch sensitive. Catastrophic failures of the Griffith type can however occur whatever the absolute value of R. Looked at in this way, ideal brittle fracture, discontinuous ductile-cleavage, mode I plane-stress necking, mode III ductile tearing and the forty-five degree shear mode in steel plates involving sliding off and plastic expansion of holes in the shear zone, are all mechanically similar, differing only in their scale [13, 26]. The fully ductile fast fracture of a large structure may thus be macroscopically brittle but microscopically ductile. When the material is not in a notch brittle condition, we have to consider in detail the spread of the fracture through plastic material and the simple decoupling of G and R to obtain the energy balance implicit in the Griffith analysis can no longer be applied [27]. In fact, continuum theory tells us that, if the flow stress is bounded at large strains, all the energy released by continuous crack advance is absorbed in plastic work [21, 23, 27–33]. We shall not dwell on the further complications arising in the characterization of fracture in the presence of considerable plastic flow [5, 27, 32–35]. This topic is still a subject of current research and there is as yet no generally agreed procedure for dealing with it [5, 32].

Fracture under these conditions is nevertheless particularly relevant to one of our important themes here – the machining of ductile materials. In this there are the added complications of very large deformations and heat flows so that variations of properties with strain, strain rate and temperature are important. We shall not attempt to discuss this topic at any length. One point is perhaps of interest however, because it illustrates how common threads may link very different phenomena. Many aspects of metal machining have been analysed with success by treating the work material as a continuum undergoing plastic deformation. Important effects caused by the behaviour of inhomogeneities in it are, however, well known [36]. These inhomogeneities may be natural or accidental second phases or particles produced by deliberate additions as in free-cutting materials [37]. On the primary shear plane these particles, through deformation and alignment, or both, promote fracture and instabilities leading to segmental or discontinuous chip formation [38, 39] by the creation of microcracks or voids. In the secondary shear zone near the rake face of the tool the particles may under go extreme deformation. The situation is very complicated and the behaviour is specific to the materials and conditions, but it is clear that the inhomogeneities can exert a profound influence on the machining operation. It is thus interesting to note that the development of anisotropic structures by deformation makes possible another, rather spectacular, example of a controlled fracture used by a craftsman, namely, the cleavage of slate. It is fitting, in this year of 1979 when the University of Sheffield celebrates the centenary of Firth College, to recall that the slaty cleavage of rocks was for a time of considerable interest to Sorby, one of the most distinguished fathers of the University. He argued that it was due to the alignment of particles or the anisotropy of structure caused by prior deformation of the rock [40–42]. He showed that this deformation had occurred in various rocks by many careful observations (of contorted beds, "green spots" in slate, oolites and encrinite joints in limestone). Moreover, he demonstrated his views by dispersing flakes of iron oxide in pipe clay, subjecting the clay to compression and baking it. He then made polished sections to show that the flakes were aligned as anticipated, and was also able to cleave his specimen into thin flat pieces on the expected cleavage plane.

The plastic deformation and rotation of embedded inhomogeneities of general shape must be treated by numerical methods. However, for viscous materials, the problem of the finite deformation of an ellipsoidal inhomogeneity of differing viscosity can be reduced to manageable proportions by using the analogy with elasticity [43–47]. The viscous problem is important in the theory of suspensions, the manufacture of glass and the measurement of strain in rocks. Structural anisotropy, whether engendered by deformation or not, may also be used to make things more resistant to fracture; examples are to be seen in the products wrought by the blacksmith and the forger, and in the science of composite materials.

Before closing these introductory remarks we note that Equation 1 relating the stress intensity factor K_1 and the crack extension force, G, first established by Irwin [48], means that when non-elastic processes near the crack tip occur within a K-dominated field a criterion for crack advance may be based either on G or K. When it can be used however, K has a commanding advantage because Ks from different loadings can be added; that is, K offers a linear characterization. Fracture mechanics proceeds in the first place by ignoring all details of the processes associated with crack advance, assuming only that the near-tip field can be characterized by K and that this field will determine all these processes, other conditions being equal. The safety of a structure is thus assured if the K values for all cracks in it are less than the critical K value for crack advance in the material, determined under similar conditions and with suitable precautions in laboratory tests. The critical K value, K_{IC} for plane strain tensile loading normal to the crack, is called the (plane strain) fracture toughness, from the physical point of view it can be regarded as a way of specifying the fracture energy under these conditions in unfamiliar units (ksi in.$^{1/2}$ or MN m$^{-3/2}$). This concept can be justified and applied successfully in linear elastic fracture and under conditions of small scale non-elastic processes. Current developments attempt to extend it by using one or more characterizing parameters differing from K, particularly those related to the crack opening displacement (COD) and to path independent integrals.

2. Path independent integrals

The formal definition of G is as follows. We let a crack tip advance by $\delta\xi$ and write $\delta E^{\mathrm{TOT}} = \delta E^{\mathrm{EL}} + \delta E^{\mathrm{POT}}$ where δE^{EL} is the increase in the elastic energy of the cracked body and δE^{POT} the increase in the potential energy of the loading system. Then $\delta E^{\mathrm{TOT}} = -G\delta\xi$. The division between body and loading system is somewhat arbitrary. For example, if we draw a surface S round the tip we can take the energy entering S from the surroundings as $-\delta E^{\mathrm{POT}}$, while δE^{EL} is the increase in energy stored within S. $G\delta\xi$ is that available at the tip to drive the crack. The crack may be regarded as an elastic singularity or inhomogeneity or as an array of crack dislocations [23, 49]. It was shown by Eshelby in 1951 [50] that if all elastic singularities and inhomogeneities within a surface Σ are displaced by $\delta\xi_l$ then $\delta E^{\mathrm{TOT}} = -F_l\delta\xi_l$ where

$$F_l = \int_S P_{lj}\,\mathrm{d}S_j \tag{5}$$

and

$$P_{lj} = W\delta_{lj} - u_{i,l}p_{ij} \tag{6}$$

Here $W(u_i, u_{i,j}, X_i)$ is the elastic energy density of the elastic (displacement) field u_i and $p_{ij} = \partial W/\partial u_{i,j}$. The result is valid for the finite deformation of a non-linear material if p_{ij} is the (asymmetric) first Piola–Kirchhoff stress tensor, X_i the initial coordinates and S a surface in the undeformed body [51–56]. The treatment can also be readily extended to generalized continua [54, 55]. The importance of Equation 5 is that we can show that

$$\frac{\partial P_{lj}}{\partial X_j} = \left(\frac{\partial W}{\partial X_l}\right)_{\mathrm{exp}} \tag{7}$$

where "exp" is the explicit derivative, that is, with u_i, $u_{i,j}$ and $X_j, j \neq l$ constant. Thus the integral is independent of the surface S if the material is homogeneous. As discussed in recent reviews [5, 21, 24], with $l = 1$ and $\mathrm{d}S_j = n_j\mathrm{d}s$ for $j = 1, 2$ we get for the crack extension force [23, 57–59]

$$G = F_1 = \int_\Gamma (W\delta_{1j} - p_{ij}u_{i,1})n_j\mathrm{d}s \tag{8}$$

F_1 is path-independent if $(\partial W/\partial X_1)_{\mathrm{exp}} = 0$; that is, if the material is homogeneous in the direction of crack extension. Thus, for example, it can be applied to a crack along an interface between dissimilar materials [54], enabling us to define the crack extension force without having to handle the the peculiar singularities which appear at the tips of cracks of this kind. The integral in Equation 8 may be transformed into several forms, some of which are valid also for finite deformation [50, 52, 60]. The integral J of Rice [57, 58] has the same form as F_1 but W is replaced by W', the density of stress-working. J and F_1 are identical if W' is independent of the strain-path or appears so in the actual deformation so that we cannot tell that an energy density function does not exist [5, 52, 62, 63]. If plastic flow has occurred at the crack tip and S lies outside the plastic region, F_1 (or J) gives the force on the crack tip plus the force on all dislocations inside S [50]. F_1 is not defined for paths within the plastic region but an integral

$$Q_l = \int_S (W\delta_{lj} - p_{ij}\beta^E_{li})\mathrm{d}S_j \tag{9}$$

where β^E_{li} is the elastic distortion tensor giving the spatial increment of elastic displacement $\mathrm{d}u^E_i = \mathrm{d}x_l\beta^E_{li}$ in a continuous distribution of dislocations [64–66], may be evaluated there and gives the resultant force on all dislocations within S [21, 67, 68].

F_1 is a path-independent integral giving the crack extension force. It is only one of a number of path-independent integrals associated with the elastic field. The general theory of such integrals is well known in theoretical physics and goes back to a famous theorem of Noether [69]. Quantities with vanishing divergence and so path-independent integrals, arise for any field when the Langrangian density function ($-W$ in the elastic case) is invariant under the operations of a continuous group, and the general consequences have been treated in a number of papers [50, 52, 54, 55] Günther [70] first applied Noether's theorem tc elastostatics, finding in addition to F_l the integrals

$$L_{kl} = \int_S (X_kP_{lj} - X_lP_{kj} + u_kp_{lj} - u_lp_{kj})\mathrm{d}S_j \tag{10}$$

and

$$M = \int_S (X_lP_{lj} - \tfrac{1}{2}u_lp_{lj})\mathrm{d}S_j \tag{11}$$

also given and interpreted by Budiansky and Rice [71]. F_l, L_{kl} and M are path-independent because a picture of a general elastic field remains one after it has been respectively translated, rotated or dilated. The conditions which must be imposed on the material and deformation to ensure path-independence of the various integrals can readily be determined from this fact [54, 57]. It has been shown [72] that these integrals F_l, L_{kl} and M are the only ones of Noether's type and that the only new feature in plane deformation is that M becomes

$$M = \int_S X_lP_{lj}\mathrm{d}S_j \tag{12}$$

a transformation which results from Gauss's theorem [54]. Several other infinite classes of path-independent integrals in two dimensions have however been found [54].

The above discussion has concerned only static cracks. An account of the force on a moving crack can be given as follows [52,73, 74]. Let S be a surface drawn round the crack tip and moving with its instantaneous velocity. The rate of increase of energy $\dot{e}(S)$ stored within S plus the rate vG at which energy flows to the tip must equal the total inward rate of energy flow $I(S)$ from the surroundings.

$$I(S) = vG + \dot{e}(S) \tag{13}$$

As the crack moves, potential and kinetic energy is added to the interior of S at the leading boundary and subtracted from it at the trailing boundary. $I(S)$ is the difference between those two rates of energy transport plus the rate of working on the material inside S by the surroundings, $\int_S p_{ij}\dot{u}_i\mathrm{d}S_j$. For a crack moving in the x_1 direction

$$I(S) = \int_S \{p_{ij}\dot{u}_i + v(W+T)\delta_{1j}\}\mathrm{d}S_j \quad \cancel{(12)}$$

where T is the kinetic energy density. If the dynamic field is a special one which moves rigidly with the crack tip we have

$$u_i(x_1, x_2, t) = u^0_i(x_1 - vt, x_2) \tag{15}$$

In this case $\dot{u}_i = -v\partial u_i/\partial x_1$ and $\dot{e}(S) = 0$ so that we can write G as

$$G = \int_S H_{1j}\mathrm{d}S_j \tag{16}$$

where

$$H_{1j} = (W+T)\delta_{1j} - p_{ij}\frac{\partial u_i}{\partial x_1} \tag{17}$$

The divergence $\partial H_{1j}/\partial x_j = 0$, so that Equation 16 is a path-independent integral for G. For a general dynamic field it is not possible to find such an integral [73]. The best that can be done is to write the field in the form

$$u_i = u^0_i(x_1 - vt, x_2) + u'_i(x_1, x_2, t) \tag{18}$$

and to try to arrange that $u'_i \ll u^0_i$ near the tip. If so,

$$G = \lim_{S\to 0}\int_S H_{1j}dS_j \tag{19}$$

Alternatively, if we can show that

$$\lim\,[\dot{e}(S)/I(S)] = 0 \tag{20}$$

as $S \to 0$, then we have

$$vG = \lim_{S\to 0} I(S) \tag{21}$$

with $I(S)$ given by Equation 14. The reader is referred to the original paper [73] for further discussion. It is verified there that Equation 21 applied to the solution for a uniformly expanding crack in plane strain [75, 76] gives a G vanishing at the Rayleigh velocity. In anti-plane strain G vanishes at the shear velocity [77]. It should also be noted that H_{lj} is not the dynamic 4×4 energy-momentum tensor of the elastic field P_{lj}. The integral of the dynamic P_{lj} gives the force on the crack tip plus the rate of change of "quasi-momentum" inside S [52].

3. Indentation fracture

In many processes where fracture is put to use, such as crushing, cutting, rock drilling, chipping and abrasive grinding, some tool or other body is forced against the object to be broken and fracture begins near the point of contact. An understanding of contact stresses [78] and indentation fracture is thus essential in thinking about these processes, as it is also when we consider related phenomena such as friction and wear which we may wish to control or prevent.

The indentation process may be approximately two-dimensional as in cutting with a knife or driving a wedge, or three-dimensional as with a hardness test. The indenter may be a blunt punch or sphere, or it may be a knife or needle. It may move normally towards the surface or be dragged over it with friction [79, 80] as in scratching, abrasion and wear. The indenter may have the sharper profile, or it may not, as when a platen bears on an asperity. The process may be slow or involve high velocity impact with a projectile. The indenter may be a liquid jet. A selection of papers [81–128] testifies to the continuing interest in many of these problems and to the variety and complexity of the phenomena observed. A recent review [81] surveys some of these, including the effects of loading speed [84–88]; of sharp and blunt indenters [81, 89–92]; and of plastic deformation in loading and unloading and in static and sliding contact [85, 86, 89, 93–105].

One of the classical problems is the formation of the ring and cone crack in the stress field under a spherical (or other blunt) indenter [84, 86, 88, 107–128]. Despite much work there is still debate about the fundamental understanding of the interesting phenomena involved. Even when the loading is idealized to a point force and a well-formed cone crack considered in a linear elastic material, only approximate or numerical solutions for G are available [113, 128]. It is perhaps therefore of interest to note that an exact solution of the analogous two-dimensional problem may be obtained [54] using the M integral of Equation 12.

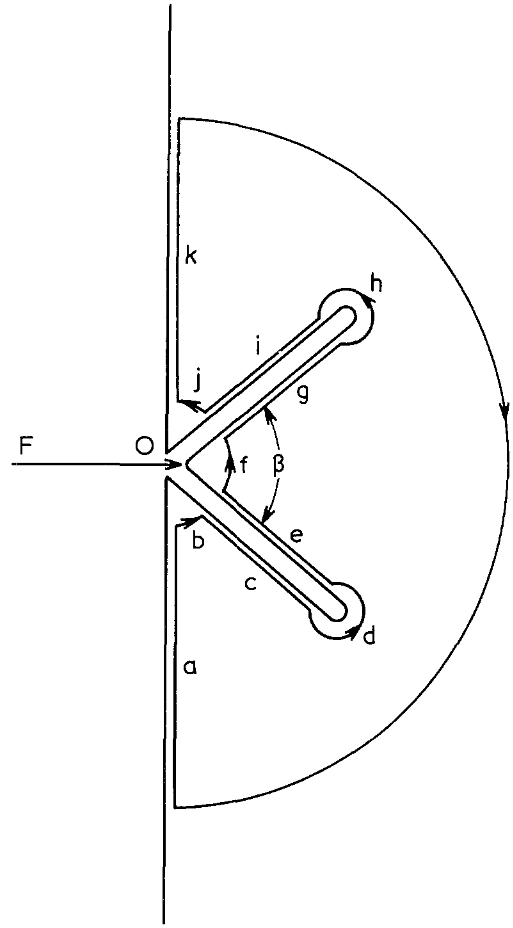

Fig. 1.

Consider this integral evaluated round the circuit indicated in a semi-infinite body containing two cracks of length a running from the origin O and inclined at an angle β (Fig. 1). There are no contributions from the portions a, c, e, g, i, k and the parts d and h together give $2Ga$. In the limit the contributions from b and j also vanish and we have

$$M = \int_{f+l} x_l P_{lj} \mathrm{d}S_j + 2Ga = 0$$

To evaluate the contributions from f and l it is sufficient to use the solution for a symmetrically loaded infinite wedge of angle γ with

$$p_{rr} = \frac{-2F}{\gamma + \sin\gamma} \frac{\cos\phi}{r}$$

The result is (there is a factor of one half missing from the previous expression [54])

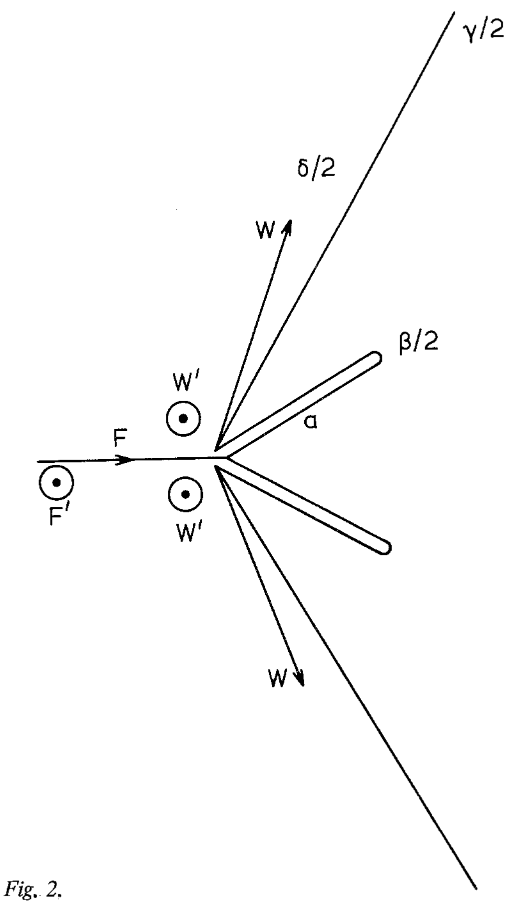

Fig. 2.

$$G = \frac{(1-\nu)F^2}{4\mu a}\left(\frac{1}{\beta + \sin\beta} - \frac{1}{\pi}\right) \qquad (22)$$

where the second term comes from the large circle l.

It is not difficult [67] to extend Eshelby's analysis to the situation of Fig. 2, which allows us to discuss the general symmetrical loading of an asperity or depression, including loads F', W' parallel to the line of the cracks. We have

$$\begin{aligned} 2Ga = {} & M[F + 2W\cos(\delta/2), 0, F' + 2W'; \gamma] \\ & - M(F, 0, F'; \beta) \\ & - 2M(W\cos\eta, W\sin\eta, W'; \zeta) \qquad (23) \end{aligned}$$

where $4\eta = 2\delta - \gamma - \beta$, $2\zeta = \gamma - \beta$, and

$$\begin{aligned} M(F, W, W'; \alpha) = {} & -\frac{1-\nu}{2\mu}\left(\frac{F^2}{\alpha + \sin\alpha} + \frac{W^2}{\alpha - \sin\alpha}\right) \\ & - \frac{(W')^2}{2\mu\alpha} \qquad (24) \end{aligned}$$

$M(F, W, W'; \alpha)$ is the value of M for a path C joining

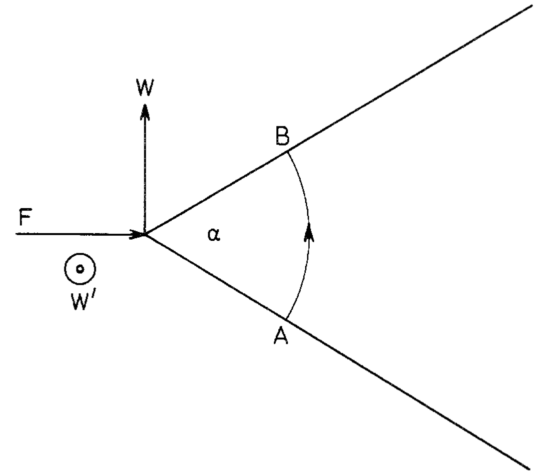

Fig. 3

any two points A and B of the faces of a wedge of throat angle α loaded by F, W and W' as indicated in Fig. 3, and extends a notation of Freund [129]. Since there is no coupling between the dashed and undashed loadings we can find K_3 from the anti-plane strain part of G,

$$\begin{aligned} K_3 & = (2\mu G_3)^{1/2} \\ & \doteq (2a\gamma)^{-1/2}\left[\left(\frac{\gamma-\beta}{\beta}\right)^{1/2} F' - 2\left(\frac{\beta}{\gamma-\beta}\right)^{1/2} W'\right] \end{aligned} \qquad (25)$$

G_3 increases without limit as $\beta \to 0$ or $\beta \to \gamma$ provided $F' \neq 0$ or $W' \neq 0$ respectively. If F' and W' have the same sign, it has a minimum $G_3(\beta^+) = 0$ at $\beta^+ = F'/(F' + 2W')$, while if F' and W' have opposite signs, it has a minimum $G_3(\beta^-) = 2F'W'/\mu a\gamma$ at $\beta^- = F'/(F' - 2W')$. The last result differs from one previously given [130, 131].

We can use the M integral in this way to find the total sum of the Ga's for any number of radiating cracks each loaded at the origin [132] and for asymmetrical loadings, but we cannot assign a G to each crack unless we can take advantage of symmetry, nor can we in general determine the separate K_1 and K_2 for plane strain loading [129].

Some other problems of interest in indentation fracture have been solved by use of the M integral [129]. The loading of a surface crack by normal [54] forces P and tangential [129] forces Q (Fig. 4) provides an example of some of the complications which may arise. The stress intensity factor is [129]

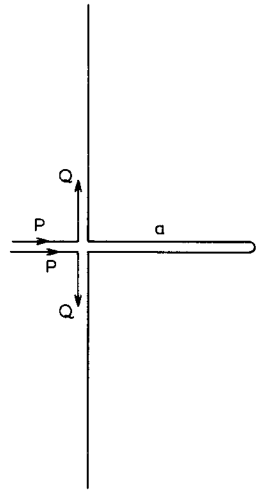

Fig. 4.

$$K_1 = \left[\frac{4\pi}{a(\pi^2-4)}\right]^{1/2}\left(Q - \frac{2P}{\pi}\right) \qquad (26)$$

This suggests that if $Q = 0$ (purely normal loading) K_1 is negative. However this is not so [61]. The effect of the forces P is to cause the two faces of the crack to impinge at the origin (as we might guess from the negative K) and so sideways forces Q are generated. In a semi-infinite body, these forces are just sufficient to annul the crack extension force – that is they are such that $Q = 2P/\pi$.

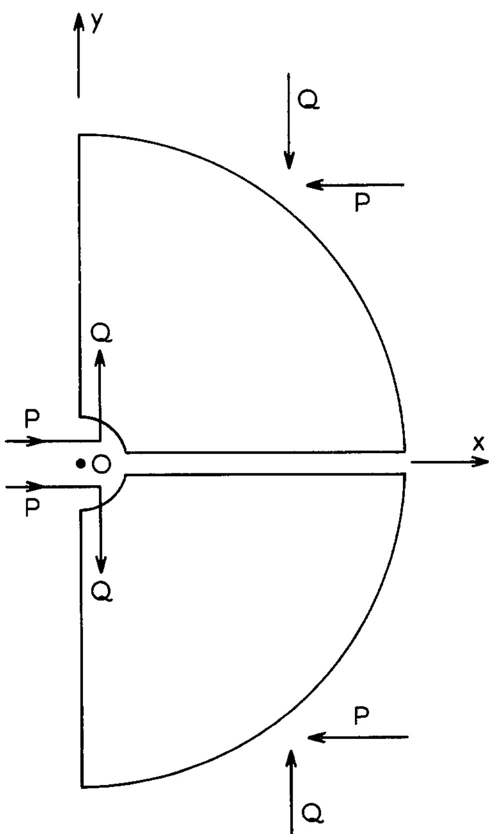

Fig. 5.

It is interesting to analyse this situation further, but care is necessary because of the infinities at the origin. A point force $2P$ applied normally (Fig. 5) to an uncracked semi infinite solid produces a simple radial distribution with $p_{rr} = -(4P/\pi)\cos\theta/r$. Evidently we may cut the body from O along Ox to any length a and retain the same stress field (and so that for a crack of any length with $K = 0$) if we apply distributed forces on the small quarter circles at the origin (and at infinity). We find easily that the resultant forces are P and Q as indicated, with $Q = 2P/\pi$. Near the origin we may write the displacements in the P and Q directions for one of the quarter planes

$$\begin{aligned} \delta_P &= \delta_{PP}P + \delta_{PQ}Q \\ \delta_Q &= \delta_{QP}P + \delta_{QQ}Q \end{aligned} \qquad (27)$$

where the singular parts of the coefficients are $\delta_{PP} \sim \delta_{QQ} \sim -(c \ln r)\cos\alpha$, $\delta_{PQ} \sim \delta_{QP} \sim (c \ln r)\sin\alpha$, with $\tan\alpha = -2/\pi$. Superposition of the two quarter planes with $Q = 2P/\pi$ gives the half plane displacements with $u \sim \ln r$ and $v =$ constant $\times$ (sgn y) on $x = 0$.

To argue generally [61] that impingement will make $K = 0$ the best we can do is to ask that $\mathrm{d}\delta_Q/\mathrm{d}a = 0$ near the origin, since δ_Q itself is infinite. With Equation 27 this gives, if we assume that the Q required depends on P but not a,

$$\frac{\mathrm{d}\delta_Q}{\mathrm{d}a} = \frac{\mathrm{d}\delta_{QP}}{\mathrm{d}a}P + \frac{\mathrm{d}\delta_{QQ}}{\mathrm{d}a}Q = 0 \qquad (28)$$

Taking $K = E^{1/2}(AP + BQ)$, where A and B are constants and writing

$$2(K^2/E)\mathrm{d}a = 2G\mathrm{d}a = 2\left(P\frac{\mathrm{d}\delta_P}{\mathrm{d}a} + Q\frac{\mathrm{d}\delta_Q}{\mathrm{d}a}\right)\mathrm{d}a$$

and using the fact that $\delta_{PQ} = \delta_{QP}$ by the reciprocal theorem, we readily find that $A^2 = \mathrm{d}\delta_{PP}/\mathrm{d}a$, $B^2 = \mathrm{d}\delta_{QQ}/\mathrm{d}a$, $AB = \mathrm{d}\delta_{PQ}/\mathrm{d}a = \mathrm{d}\delta_{QP}/\mathrm{d}a$. Now $BKE^{-1/2} = ABP + B^2Q$, and so, if Equation 28 holds, $K = 0$. We note that this argument could apply to other geometries. Thus the assumption of a zero in K of this type, together with Equation 28, would enable us to deduce the K due to one loading from that due to another [61].

Expression 26 has interesting implications for indentation fracture [129]. For some loading paths in the (P, Q) plane a reduction in P may lead to an increase in crack extension force and so to cracking on unloading (the impingement at the origin which occurs when $2P$ is applied alone means that some re-interpretation of the forces Q in a given situation may be necessary). A similar discussion could be given using Equation 23, with more general geometries. The phenomenon of the development of cracks under indentations on unloading is relevant to chipping, scratching, wear and drilling [81]. An important part of the behaviour is controlled by the residual stresses which are developed due to plastic flow under the indenter [85, 86, 89, 93–106], though lateral cracks may form without plasticity [86]. In the discussion based on Equation 26, the effect of the complicated loading under the indenter is represented [129] in a simplified way by the forces P and Q. It is evident from the opposing effects of these forces (or from the various influences of F, W, F' and W' in Equation 23) that the behaviour may be very sensitive to the local conditions near the point of application of the load. If at this point a crater of rather loosely bonded fragments has been produced, the amounts of normal and lateral forces generated on increasing or decreasing the normal load may be sensitive to the local asperities and depressions of the specimen and the indenting tool.

The theory of cracks loaded by distributions of forces is obviously a starting point for the consideration of the action of wedges and knives. Here the introduction of forces F' and W' parallel to the knife, as given in Equation 23, or in the corresponding elaboration of Equation 26 might be relevant to an idealized discussion of the effect of the sawing or dragging action used with knives. There seems to be little work published in this field of cutting with sharp knives [133], although cutting and shearing are relevant to comminution machinery [134], and pressing and impacting against sharp edges with shear and normal loading have been studied [135, 136].

4. Comminution

In comminution we reduce the size of finite bodies by repeatedly separating them into pieces. As with most large human endeavours which are long-established its practice is very complicated. A recent survey [133] lists five techniques currently of commercial significance – explosive shattering, electro-hydraulic crushing, ultra-sonics, mechanical means and the use of jet or fluid energy mills. It classifies machines commonly in use according to the dominant process in them – crushing, impact or attrition (abrasion). As with all fracture processes the environment influences the behaviour and it is sometimes necessary to take into account balancing processes of agglomeration which occur as the particles become finer [137, 138]. Important from the practical point of view are the efficiency of the process [133], which can be assessed in several ways [139], and the size distribution of the particles produced. There are a number of theories of the comminution process, expressing these and other indexes in terms of more fundamental physical parameters describing the method of size reduction used and the materials involved [133, 140–142]. Moreover, with the advent of increasing computing power, the number and complexity of theoretical discussions of the effect of the repeated recycling involved in milling processes have increased considerably. The state of the theory of these difficult questions, so important in practice, are treated by experts elsewhere at this meeting and we shall not attempt to discuss them.

Generally in practical processes a particle is crushed and the pieces re-crushed repeatedly. Many investigators of the fundamental parameters controlling the comminution process and the size distribution it produces have thus gravitated towards studies of what happens when a particle or specimen is crushed once – single particle crushing. There have been many investigations of this topic, using both impact and slow crushing, some including most elegant studies using high speed photography [136, 143–157]. As in indentation fracture one has to consider the effect on the cracking process of the size, shape and properties both of the particles and the indenters or platens, and the methods of applying the load, including particularly its speed and whether the relative velocity at contact is purely normal or has also a tangential component. In the compression of spheres two distinct modes of failure have been noted [143, 148, 151]; ring and cone cracking under the points of application of the load may lead to initial separation into pieces like the peelings of an onion, while greater deformation under these points of contact may cause a wedging action which splits the specimen into segments like those of an orange. Depending on the conditions and the properties of the bodies, the separation may be a relatively quiet one or an explosive shattering, and may be into a few pieces or many. We shall later consider these conditions and properties more carefully, but we begin by examining a special shattering process, contrived to ensure that the body breaks safely into many pieces.

5. Energy relations in crack dynamics, comminution and shattering

The shattering process we discuss is that of the breaking of toughened glass as in the all too familiar example of the car windscreen. A toughened glass plate has a compressive stress in its surface layers while its interior is in a state of balancing tension. Cracks do not run readily through the surface layer, but when they do reach the interior, catastrophic failure takes place. A very rough calculation shows that the stored elastic energy is just about equal to the minimum energy required to fracture the plate so that, in contrast to the usual grinding operation, this comminution process appears to be rather efficient. A typical value [158–161] for the fracture energy R of a glass plate is 5 to $10\,\mathrm{J\,m^{-2}}$, while for the residual tensile stress σ_T in a thermally toughened plate we take [158] $10\,\mathrm{MN\,m^{-2}}$. Then the stored elastic energy per square metre, $(1-\nu)(\sigma_T^2/E)d$, of a plate of thickness d is equal to the work $(2R/x)d$ needed to shatter it into pieces of size x when x is 1 to 2 cm, a result of the right order of magnitude. Moreover, $\sigma_T^2 x$ should be constant, as observed [159]. Of course we have neglected the elastic energy in the fractured pieces and the fact that the surfaces are in compression [159], but the conclusion that most of the elastic energy is used in supplying the fracture energy is in accord with the motorist's usually fortunate experience that his screen shatters quietly with little energy to spare. The reason for the apparent efficiency of this comminution is that we have enlisted the aid of the forces arising from the permanent deformation combined with thermal expansion to generate the internal stress without saying what this process has cost us in thermal energy. The shattering of rocks by decrepitation due to thermal shock is a well-known and very ancient art, but when a true energy balance is struck, its potential for comminution seems to be doubtful [133, 162], though heat is used in some methods of rock drilling [133, 163, 164]. There is evidence of more advantage however when the temperature changes include those of phase transformations causing internal stress and microcracking in the material undergoing comminution [162]. We should thus perhaps always remain alert to any possibility of using thermal or chemical forces in comminution, since these are very large compared to those which we can apply by any conventional mechanical means.

More detailed theories of the shattering of toughened glass have been constructed by estimating the critical G needed for crack forking and so the particle size from a mean free path for repeated crack division [159, 165]. Again the relation $\sigma_T^2 x = \text{constant}$ is obtained [159]. The motion, paths, deviations and forkings of cracks have been much studied [59, 165–230], especially, at high crack speeds, by the use of sophisticated photographic techniques. Fig. 6 shows a high speed photograph by Field [166] of the failure of toughened glass, in which an initial regular forking is clearly seen to be modified to produce the familiar final craze pattern by a reflected stress wave [188, 189] returning from the bottom of the specimen. Fig. 7 shows some previously unpublished pictures by Soltész [190] of the initial forking, with a picture of the behaviour in untoughened glass for comparison.

The path that a crack will follow and when it will turn or fork are topics of importance to the fundamental understanding of comminution, shattering processes and indentation fracture generally. We refer only briefly to the much studied question of the possible criteria for the crack path. The subtlety here is that as a crack advances, turns or forks, it continually alters the field in which it moves. A possible criterion [61] is that the crack moves so that the quantity F_2 given by Equation 5 is always zero. Since we may show that $F_2 = -2K_1K_2$, [191, 192], this criterion includes another possible alternative, namely that the motion is such that $K_2 = 0$, [193, 194], or that the crack moves to maintain a symmetric field at its tip [59, 195]. The integral L_{KI} of Equation 10 may be used to interpret F_2. We can show that if a crack has a deviation at its tip forming a kink at an angle α, then $f_2 = [\partial f_1/\partial\alpha]_{\alpha=0}$ where f_1 and f_2 are the values of Equation 5 evaluated round the tip of the kink [55]. Thus $f_2\,\mathrm{d}\alpha$ is the change in the crack extension force caused by a small deviation $\mathrm{d}\alpha$ of the main crack; there is no

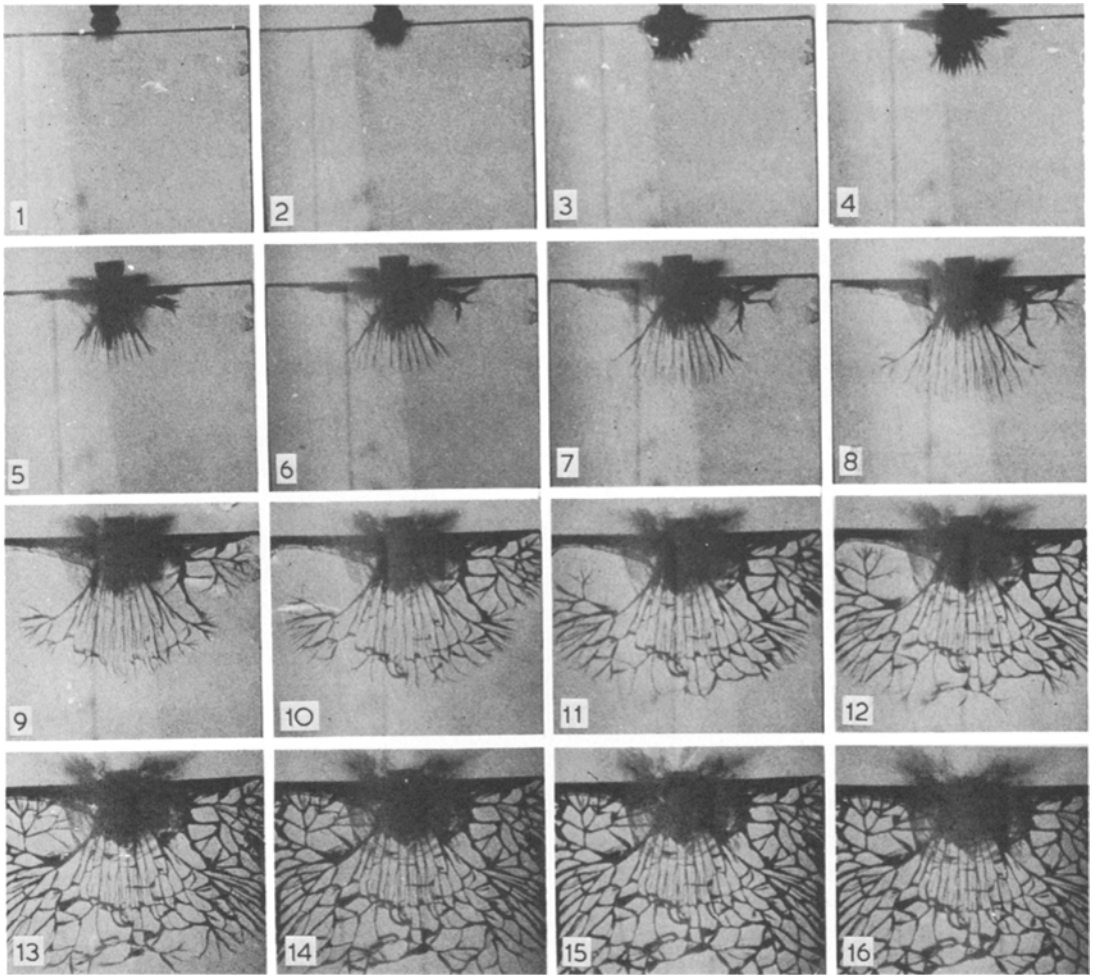

Fig. 6.

first order change in f_1 if $f_2 = 0$. Some problems in the theory of crack paths, and the use of formulae for the stress intensity factors at forked and kinked cracks to discuss them, have been briefly considered elsewhere [5, 24, 196–199, 268]. The analysis of the kinked crack [196] has also been applied [200, 201] to discuss crack stability [202–204].

The energy relations at the tips of cracks which are moving or forking are subjects which bristle with subtleties [74], many of which are brought out clearly in recent papers by Rose [205–208]. To discuss the motion of a crack tip we must let it move in an arbitrary manner, so that the tip is at $x = \xi(t)$, say, and calculate the energy available for its propagation. The calculation of this energy release rate G for arbitrary (rectilinear) motion was first carried out for anti-plane strain by Kostrov [209] and Eshelby [210, 211] and later extended to plane strain conditions by Freund [212–215]. It turns out that G depends on ξ and $\dot{\xi}$, but not $\ddot{\xi}$; the crack tip thus behaves as if it had no inertia [210, 211, 216, 217]. If the fracture energy R is also assumed to depend on ξ and $\dot{\xi}$, we then have the equation of motion [210, 211]

$$G(\xi, \dot{\xi}) = R(\xi, \dot{\xi}) \tag{29}$$

The expression for G may be written

$$G = g(v)G^* \tag{30}$$

where G^* depends on the current length of crack, the applied load and the history of crack extension, but not on the instantaneous crack speed v [210–215]. For plane strain $g(v)$ can be approximated for most practical purposes by the linear relation [205]

$$g(v) = 1 - v/c_R \tag{31}$$

where c_R is the Rayleigh velocity.

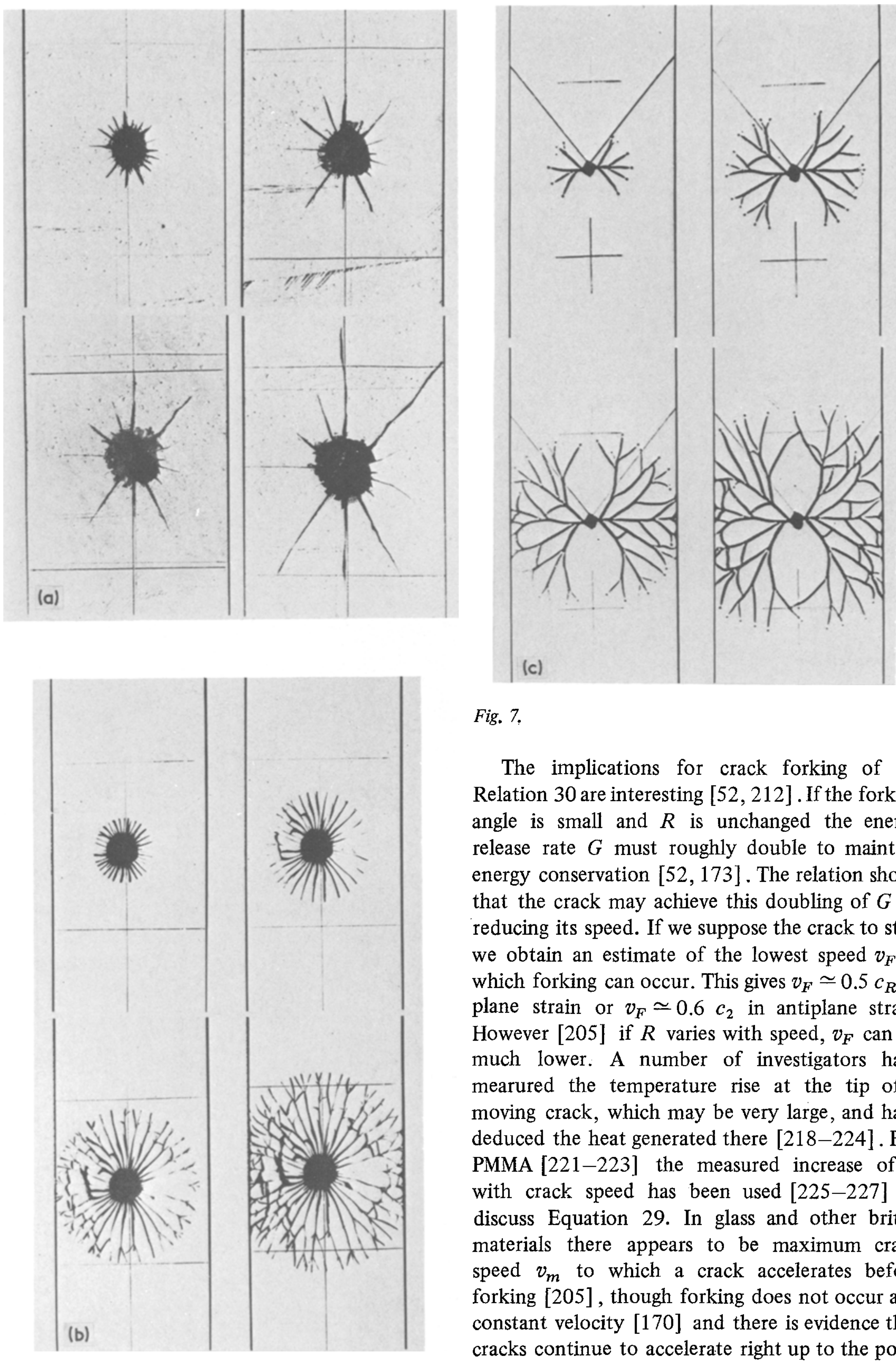

Fig. 7.

The implications for crack forking of the Relation 30 are interesting [52, 212]. If the forking angle is small and R is unchanged the energy release rate G must roughly double to maintain energy conservation [52, 173]. The relation shows that the crack may achieve this doubling of G by reducing its speed. If we suppose the crack to stop we obtain an estimate of the lowest speed v_F at which forking can occur. This gives $v_F \simeq 0.5\ c_R$ in plane strain or $v_F \simeq 0.6\ c_2$ in antiplane strain. However [205] if R varies with speed, v_F can be much lower. A number of investigators have mearured the temperature rise at the tip of a moving crack, which may be very large, and have deduced the heat generated there [218–224]. For PMMA [221–223] the measured increase of R with crack speed has been used [225–227] to discuss Equation 29. In glass and other brittle materials there appears to be maximum crack speed v_m to which a crack accelerates before forking [205], though forking does not occur at a constant velocity [170] and there is evidence that cracks continue to accelerate right up to the point of forking [179]. The situation is complicated and not completely understood [205]. It is clear that as the crack grows the factor G^* increases and that the energy balance Equation 29 links variations in $g(v)$ and $R(\xi, \dot{\xi})$. Repeated forking is favoured when the crack begins in an "overloaded" condition, for example, from a blunt rather than a sharp notch [166, 167, 179] or from a knife edge struck with increasing load [167]. The crack first maintains the energy balance by accelerating to reduce $g(v)$ and increasing R by generating more dissipative processes at its tip. G^* becomes larger for a given crack length as the initiating stress field becomes higher [168], so that forking to enable the effective R to increase begins earlier and subsequently occurs more frequently. One might perhaps expect that the forking in materials showing a marked increase in energy dissipation with crack speed would differ from that shown by materials which do not. Forking is certainly not precluded in materials showing dissipation [177, 221, 223]; indeed it has been shown directly that [223] for PMMA the sum of the heat generated in the two branches after forking is equal to that generated at the tip before the forking occurs, as one would expect from the energy argument [173] It has been noted [205] that the initial crack forking angle is greater when forking occurs at shorter crack lengths and the suggestion made that this implies an influence of processes ahead of the crack tip on the forking, and that the influence of reflected waves from these on the stability may be important [166, 205]. Fig. 6 shows that stress waves reflected from the surfaces can certainly affect the crack path [166]. Indeed the influence of stress waves is the basis for the deductions made from the Wallner line [228] patterns and for the beautiful techniques using ultra-sonic waves to modulate the crack front [182, 229]. It is evident however that repeated forking may occur before interactions with waves reflected from the external surfaces become important. Moreover, the advance of the locus of the tips of the repeatedly forking cracks [167] is approximately at the limiting velocity v_m ($\sim 1500\,\mathrm{m\,sec^{-1}}$ in glass), so that there is no great reduction in crack velocity on forking [166, 230]. It is possible [61] that this is because the change in velocity necessary to double the $g(v)$ factor becomes progressively smaller as the crack speed increases and $g(v)$ itself becomes smaller.

In the fracture of toughened glass we ensure that the cracking starts in an "overloaded" solid, but with a carefully controlled excess of energy available. In other shattering processes, as when we use a hammer to crack a nut, this control is lacking. The failure of a brittle solid when crushed is often a catastrophic shattering because much energy is stored in it before fracture begins. This situation will be accentuated when the energy is stored very rapidly and when the field is inhomogeneous, and the site where fracture may start most easily does not coincide with the site of maximum stress. Once fracture does begin somewhere, of course, there will be a rapid redistribution of stress by stress waves and a spreading and initiating of fracture throughout, as in the repeated forking with stress-wave interaction already discussed.

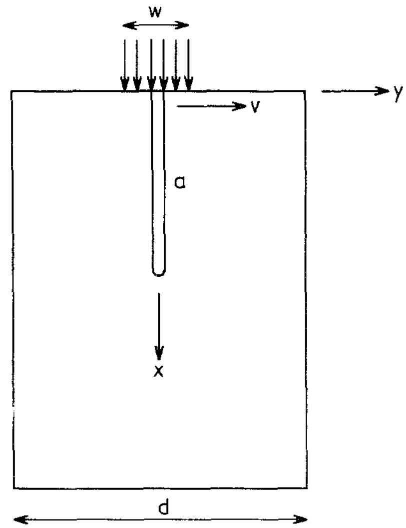

Fig. 8.

To study compression failure in a controlled way we may deliberately introduce a starting crack before the test begins [231–233]. A platen applies a force density F/w over a width w to a specimen of width d and thickness b normal to the paper containing a vertical crack of length a (Fig. 8). We use simple beam theory to derive the crack extension force. Each half of the specimen is subject to a couple $(F/8)(d-w)$ and, with the y displacement v taken to be zero at $x = 0$ and $x = a$, has an energy $a(F/8)^2(d-w)^2/2EI$, where $I = bd^3/96$ (there is also energy due to the uniform compression, but this does not depend on a). If the crack advances δa under constant load, then, by the general theorem that $\delta E^{\mathrm{POT}} = -2\delta E^{\mathrm{EL}}$, $G\delta a = \delta E^{\mathrm{EL}}$. Thus

$$G = \frac{3F^2}{2Eb^2 d}\left(1 - \frac{w}{d}\right)^2 \tag{32}$$

A force F_s for splitting is determined by setting $G = R$. If the width w is governed by yielding under the platen, the force for yielding is roughly

$$F_y = Ybw \tag{33}$$

where Y is the yield stress. The condition $F_s = F_y$ determines a critical size of particle below which failure will occur by plastic squashing rather than by fracture. Thus the well-known intervention of plastic behaviour in the comminution of fine particles [137, 149] is explained [232, 233].

Kendall's experiments on polystyrene fit his theory well; he uses similar methods to discuss the effect of platen size ($w < d$), crack stability and lateral pressure [232]. The analysis is approximate and the possible development of sideways forces Q (see Section 3) is not considered (the boundary conditions keep $v = 0$ at $x = 0$, while the crack tip – which is not accurately represented by the simple beam theory – opens at a finite angle). However, it is difficult to see how the theory could be improved within the confines of the simple beam approximation. For example, allowance for the development of sideways forces Q (as in Section 3) and imposition of an additional condition, say, $dv/dx = 0$ at $x = a$ (also not a proper representation of a singular crack tip), reduces the G of Equation 32 by a factor of four. (In contrast to the situation in the semi-infinite body considered in Section 3 it does not make G zero.) The R values that follow are then rather low. More accurate formulae for G might be obtained for specimens with a shape which is easier to analyse; for example, some progress could be made with cracked circular cylinders [234, 235].

In modern windscreens the extent of the toughening is often deliberately reduced in some regions to give larger particles and better visibility through the windscreen fragments [166]. We might hope that the theory of this process would shed some light on how we should approach the much discussed questions of the size distribution and efficiency of the comminution processes, at least for single particle crushing, although the shattering of toughened glass must become less and less relevant as the comminution being studied increases in violence. However, as we have tried to indicate, these problems are very complicated and we shall not attempt at this time to make any detailed assessment of the theoretical basis of the ideas of Kick [236, 237], Rittinger [238], Bond [239–241], Holmes [242], Charles [243, 244], Rose [245] and of the many others who have contributed to these important topics, or to add to the debate which has raged about them [133, 139–142].

Snow and Paulding [150] who review some earlier discussions of the problem, have given a theory of the size distribution to be expected when an uncracked sphere is crushed between opposite point forces, estimating the energy density from the known elastic solutions [246–249] (there are problems, because the energy in regions near the forces is infinite). They find however that the observed size distribution is better predicted by assuming that the surface area S_0 produced by sub-dividing an element is proportional not to the energy E_0 stored in it, but to $E_0^{1/2}$. We should not expect an exact proportionality, of course, because some of the elastic energy is going elsewhere. It is interesting that this assumption also fits approximately the relation found by Kerkhof [165] between the number of segments of a glass disc fractured by a point load, and the stored energy in it.

A rough estimate of the efficiency of the breakage process in which ring and cone cracks spread from the opposite poles of a sphere between platens [143] has been given [81]. From the Hertz theory the work W_L required to press an (uncracked) particle of radius a against a platen to a load P is

$$W_L = \frac{2}{5}\left(\frac{4k}{E_1}\right)^{2/3}\left(\frac{P^5}{a}\right)^{1/3} \tag{34}$$

where $k = (9/16)[E_2(1-v_1^2) + E_1(1-v_2^2)]/E_2$ and the suffix 1 refers to the particle (if we consider two particles of radii a and b, a is replaced by $2ab/(a+b)$). We assume [81], guided by the observations, that the relation $P(\alpha)$ between the load P and the approach α of the platen and particle is insensitive to the appearance of the cone crack, so that Equation 34 is taken to hold during the whole process of driving the crack through the particle. P is estimated by using the expression of Roesler [113] for the G of a conical crack, taking the radius of the skirt equal to that of the sphere; this gives $P^2 = REa^3/\kappa(\nu)$ where (from theory or experiments) $\kappa(\nu) \sim 10^{-3}$. The energy U_s required for fracture is at least the area of the cone crack time R the fracture energy (it will be greater if the sphere breaks into many pieces or if the fragments acquire kinetic energy). Taking $U_s = \pi a^2 R/\sin\alpha$, where α is the semi-angle of the cone, the efficiency $\eta = U_s/W_L$ is 0.03 for $a = 100\,\mu$m and typical values of the other parameters for glass [81]. It is noted that η is greatest if the indenter (platen, other particle) is rigid. To refine this calculation we need to evaluate W_L by using formulae appropriate to cracked finite bodies, and also to construct a theory which estimates the particle size distribution of the fragments so that U_s can be suitably modified. We note also that P has been fixed by the quasi-static equilibrium of the cone crack for a particular skirt radius (equal to a). Thus the questions of the initiation of cracking and of the possibility of overloading because the site of maximum stress does not coincide with that where fracture can start most easily (the "weaker flaw"), are not considered. It is interesting to estimate W_L independently by using experimental results. For glass spheres with $2a = 3.05 \times 10^{-3}$ m, P is roughly normally distributed [245] with a maximum value of 200 lbwt = 894 N; this gives $W_L = 0.173$ J, a value much greater than the least energy $U_s \sim a^2 R = 10^{-5}$ J needed for fracture. We can thus see how the possibility of shattering arises and how important the detailed analysis of crack initiation and propagation with the correct finite body geometry is in the discussion of the comminution efficiency.

6. Flint knapping

In crushing, grinding and in the drilling and chipping of brittle materials a component of the fracture process is often the removal of a thin piece or shaving from the surface. This is also the essence of the ancient craft of flint knapping, doubtless the oldest example of man's controlled use of fracture for his own ends. In pressure flaking the piece is removed by a steady pressure while in percussion flaking a sharp blow is used. In both of these processes the external force is highly localized and the crack moves along very near a free surface. It is thus greatlv influenced by repeated reflections of the disturbance it creates. Indeed this influence must be a general phenomenon in the fracture of small particles. Another example where it might be interesting to examine the transmission of energy to the crack tip is the function of glass-cutter's light tap in persuading an unwilling crack to run along his scratch. Again, those interested in making tumblers, dishes, candlesticks and other ornaments from old bottles may buy a fascinating little apparatus which

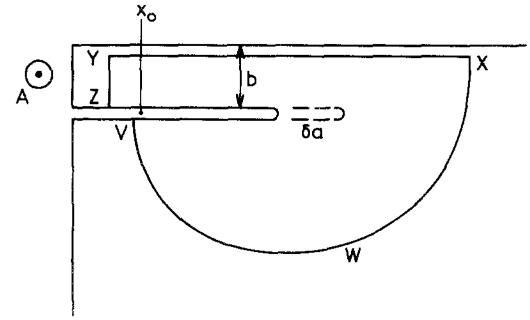

Fig. 9.

enables any careful child to produce controlled perfect circumferential fractures as easily as a chemist breaks glass tubing. An external circumferential scratch is first made. This is then lightly tapped from inside with a small hammer consisting of a circular metal disc with a tapered edge, attached to the end of a long rod. The crack usually advances in small steps, but sometimes, particularly towards the end of the process, in larger ones. Over zealous use of the hammer causes deviations.

Some studies of the fracture mechanics of flint knapping [127, 250] and of some wavy conical fractures apparently made by ancient man [251, 252] have been carried out. Here we discuss briefly the flaking processes, basing our treatment on that of Atkinson *et al.* [250] and on an earlier review [74]. We use a very idealised geometry to make the analysis tractable, but nevertheless are able to bring out some interesting points about the efficiency of the processes involved.

We can give an approximate treatment of pressure flaking by the same type of argument used to discuss the splitting of a single particle (Section 4). For simplicity we use anti-plane strain, so that (Fig. 9) a force parallel to $+z$ is applied upwards from the paper at A. Then if x_0 is a distance behind the crack tip several times the flake thickness b there is a uniform stress $p_{xz} = -\sigma$, say, in the flake when $x < x_0$, and the elastic energy density is $\sigma^2/2\mu$. By the usual argument, when the crack length increases by δa the elastic energy (in unit thickness normal to the paper) increases by $\sigma^2 b\delta a/2\mu$ while the external force does an amount of work δV just equal to twice this. Thus

$$G = \sigma^2 b/2\mu, K = (2\mu G)^{1/2} = \sigma b^{1/2} \quad (35)$$

and the efficiency $\eta_P = G\delta a/\delta V = \frac{1}{2}$. To assess the dependence of η_P on the velocity let the crack move to the right at a speed v. There is now a kinetic energy density $T = \frac{1}{2}\rho \dot{w}^2$ where $\rho = \mu/c^2$ and $w(x, t) = -\sigma(x - vt)/\mu$ is the uniform displacement field in the flake well behind the crack tip (c is the shear wave velocity). As the crack moves $\delta a = v\delta t$, the energy increases by $(\sigma^2 b\delta a/2\mu) + (\sigma^2 v^2 b\delta a/2\mu c^2)$. In this time the external force does work $\sigma^2 b\delta a/\mu$. Thus

$$G = (\sigma^2 b/2\mu)\left(1 - \frac{v^2}{c^2}\right) \tag{36}$$

and the efficiency is

$$\eta_P = vG/(\sigma^2 vb/\mu) = \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right) \tag{37}$$

It thus decreases with velocity.

It is easy to confirm the above expressions for G by using the energy–momentum tensor, integrating round the path $S = VWXYZ$. Only YZ makes any contribution since that on VWX vanishes because it does so if $W, X \to \infty$ and is independent of S. Thus, since $p_{xj}\mathrm{d}S_j = 0$ on XY,

$$\begin{aligned} G = F_x &= \int_S \left(W\delta_{xj} - p_{zj}\frac{\partial w}{\partial x}\right)\mathrm{d}S_j \\ &= \int_{YZ}\left(W - p_{zx}\frac{\partial w}{\partial x}\right)(-\mathrm{d}y) \\ &= \sigma^2 b/2\mu \end{aligned}$$

For the dynamic case the general formula of Equation 21 becomes

$$vG = \lim_{S\to 0}\int_S \{v(W+T)\delta_{xj} + p_{zj}\dot{w}\}\,dS_j \tag{38}$$

However, this field is a special one in which $w(x, t) = w(x - vt)$ so that $\dot{w} = -v(\partial w/\partial x)$ and G is actually independent of S. Its evaluation gives at once Expression 36.

These results, obtained by approximate arguments, can be confirmed by exact solutions. For the static crack the required harmonic function w is $\mathrm{Re}(\zeta)$ where $z = x_1 + \mathrm{i}x_2 = \zeta + \mathrm{e}^{\zeta}$ (the analogous solution in hydrodynamics is the two-dimensional "Borda's mouthpiece", with w corresponding to the velocity potential ϕ). This w has the form $(\sigma/\mu)(2b/\pi r)^{1/2}\sin\frac{1}{2}\theta$ near the tip, giving $K = \sigma b^{1/2}$ in agreement with Equation 35. To obtain the dynamic solution we apply the Lorentz transformation to $w(x, y)$ replacing x by $(x - ct)/\beta$ and multiplying the result by β, where $\beta^2 = 1 - (v^2/c^2)$, to maintain the applied load at σb. Near the tip the resulting w has the form

$$w = L(v)(2\pi)^{-1/2}(vt - x)^{1/2} + O[(vt - x)^{3/2}] \tag{39}$$

where

$$L(v) = (4/\mu)(1 - v^2/c^2)^{1/4}K(0)$$

with $K(0) = \sigma b^{1/2}$, the static value. The dynamic $K(v)$ has the form $K(v) = (1 - v^2/c^2)^{3/4}K(0)$ and $G(v) = \frac{1}{8}K(v)L(v)$, which confirms Equations 36.

To discuss percussion flaking a pulse

$$w_{\mathrm{a}} = -(c/\mu)f\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right) \tag{40}$$

is applied to the flake, Here $H(\lambda) = 1$ for $\lambda > 0$ and $H(\lambda) = 0$ for $\lambda < 0$. When the pulse first hits the crack at $t = 0$, the tip is assumed to begin to move with constant velocity v. The field may be found by the Wiener–Hopf technique, G being obtained in terms of a function $Q(t)$,

$$G = 2b\mu^{-1}(1 - v^2/c^2)^{-1}[Q(t)]^2 \tag{41}$$

The Laplace transform $\bar{Q}(p)$ of $Q(t)$ is an infinite product and an explicit expression for G can only be found by inverting it. However, the energy E_{A} absorbed from the pulse by the crack tip can be obtained without this inversion by using Parseval's theorem:

$$\begin{aligned} E_{\mathrm{A}} &= \int_0^\infty vG(t)\,\mathrm{d}t \\ &= 2bv\mu^{-1}(1 - v^2/c^2)^{-1}\int_0^\infty [Q(t)]^2\,\mathrm{d}t \\ &= 2bv\mu^{-1}(1 - v^2/c^2)^{-1}(2\pi i)^{-1}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \bar{Q}(p)\bar{Q}(-p)\,\mathrm{d}p \end{aligned}$$

Finally, after some reduction,

$$E_{\mathrm{A}} = 2b(\pi\mu)^{-1}(1 + v/c)^{-1}\int_0^\infty |\mathrm{i}\lambda\bar{f}(\mathrm{i}\lambda)|^2 \exp(-\delta^{1/2}\lambda b/c)\,\mathrm{d}\lambda \tag{42}$$

Here $\delta = (1 - v/c)/(1 + v/c)$. This form is convenient for investigating the effects of different shapes of pulse $f(t - x/c)$. A uniform stress pulse $-\sigma$ of length a in space or a/c in time leads to

$$E_{\mathrm{A}} = 2ab(v/c)(1 + v/c)^{-1}(\sigma^2/\mu)h(a/b\delta^{1/2})$$

The function $h(x)$ tends to x/π for small x and to 1 when x is large while for a long pulse $a \gg b, h(x) = 1$. The crack tip interacts with the pulse for a time $a/(c-v)$ and during that time moves a distance $\bar{x} = av/(c-v)$. Hence the average energy absorbed by the tip per unit advance G_{AV} is obtained by setting

$$G_{AV} = E_A/\bar{x}$$

This gives

$$G_{AV} = (2\sigma^2 b/\mu)(1-v/c)(1+v/c)^{-1} h(a/b\delta^{1/2}) \tag{43}$$

The energy E_I injected in the pulse is half potential and half kinetic and so is $ab\sigma^2/\mu$, giving an efficiency

$$\eta_{PER} = E_A/E_I = 2(v/c)(1+v/c)^{-1} h(a/b\delta^{1/2}) \tag{44}$$

As $v \to c$, $\eta_{PER} \to 1$, in contrast to the η_P given by Equation 37 which tends to zero. This difference arises because in pressure flaking the velocity at the free end of the flake is proportional to the crack speed so that the rate of absorption of energy and vG tend to zero together, η_P remaining finite. In percussion flaking, however, the velocity imparted to the free end of the flake by the blow depends only on the mechanical impedance of the material and the injected energy is independent of the crack speed. The efficiency is thus greatest at high crack speeds when the crack absorbs most energy.

The above discussion has treated only very simple geometries. It would be interesting to try to extend the work to those that are more realistic and to attempt a more detailed interpretation of some of the phenomena revealed by the high speed photography of shattering and crushing.

7. Conclusion

One of the topics we have not so far discussed here is the use of additives to facilitate milling, grinding or rock-drilling [253, 254]. The explanation of their action is bound up with our understanding of the general problem of the influence of the environment on deformation and fracture (the theme of the third Tewksbury lecture [255]), an understanding which is still incomplete, despite much study in this field. It is natural to study the influence of the environment on the classical fracture mechanics parameters. This has led to the correlation of crack growth rates with stress intensity factors and to the idea of a threshold stress intensity factor K_{th} or K_{Iscc} for measurable crack advance in the presence of an active environment. Recent developments of interest have included a re-examination [256–259] of the influence of changes in surface energy due to adsorption [260–262]. Some novel ideas have also been introduced by Howard [263] in an attempt to relate fracture toughness to changes in surface energy. Although so far applied only to the embrittlement of steels by hydrogen, it seems worthwhile to give a brief account of this work, since the principles involved should also apply to other fractures in an active environment. The surface energy changes due to adsorption are, even in "brittle" materials, usually much smaller than the fracture energy. It has been recognized that these small energy changes may nevertheless tip the balance of behaviour from ductile to brittle [1, 28, 31, 264] but there has been little quantitative development of this idea. The energy release rate G^{Δ} associated with a *finite* crack advance Δa in an elastic–plastic material [33] affords a means of relating a fracture energy to a surface energy. In the DBCS model [34], the fracture toughness is related to G^{Δ}, Δa and the yield stress σ_y by

$$K_{Ic}^2 = 0.52\,\sigma_y^2\,\Delta a \exp\left(\frac{0.43\,EG^{\Delta}}{(1-\nu^2)\,\sigma_y^2\,\Delta a}\right) \tag{45}$$

Taking G^{Δ} to be twice the surface energy and estimating the reduction in the latter caused by adsorption of hydrogen at pressure p the relation

$$(K_{th}/K_{Ic})^2 = \exp\left[\frac{-0.86\,\Gamma_s EkT \ln(p/p_0)}{(1-\nu^2)\,\sigma_y^2\,\Delta a}\right] \tag{46}$$

for the ratio of the threshold toughness K_{th} at pressure p to the unembrittled value K_{Ic} is obtained. Here Γ_s is the saturation adsorption parameter in the expression $\Delta\gamma = -\Gamma_s kT \ln(p/p_0)$ for the reduction in surface energy [261]. The relation 46 fits recent data [265] for the embrittlement of steel by gaseous hydrogen very well while Equation 45 gives the linear correlation between $\ln(K_{Ic}/\sigma_y)$ and $1/\sigma_y^2$ found when other variables are constant [266, 267]. However the required G^{Δ} is still too large to be interpreted as a surface energy. Accordingly it is further assumed [263] that at each increment of crack growth in these (intergranular) fractures the crack front is presented with an ensemble of deviations determined by the angles θ of grain boundaries on which it may proceed. Those boundaries having θ smaller than some θ_c will cleave, contributing virtually nothing to G^{Δ}, while those with θ greater than θ_c will tear, contributing an amount G_0^{Δ} say. Thus we expect an increasing proportion of tearing as the embrittlement proceeds, as is observed. To find $E(\theta_c)$, the expected value of the statistical variable θ_c, the theory of the emission of blunting dislocation loops from a crack tip [1] is extended [263]. A crack with a kink [196] of angle θ is examined and the boundary θ_c between cleavage and spontaneous emission of a dislocation loop is determined. In this way it is found that the results on embrittlement by gaseous hydrogen [265] require that $\Delta a \sim 10^{-6}$ m and $G_0^{\Delta} \sim 550\,\mathrm{J\,m^{-2}}$. It is encouraging that these values are quite close to those derived from the different use of Equation 45 to interpret the experiments in which σ_y is varied [266, 267].

We have tried in this paper to discuss some of the fundamental questions in the mechanics and physics of fracture which may be relevant to its use as a tool in breaking or parting materials. On the important practical problems we have been able to say very little and we look forward to learning more about them at this meeting. The focus of the symposium on the controlled use of fracture to produce desired changes of shape is most timely. There are many fascinating problems associated with man's activities in this field, as there are in the converse endeavour to ensure the integrity of engineering structures. These complementary sciences will surely benefit from their mutual interaction. Making its contribution to this interaction and to the advancement of both sciences, the meeting is itself a most fitting tribute to the memory of Jack Osborn, whose name will be associated always with the Tewksbury Symposia he helped so much to establish.

Acknowledgements

I wish to thank Drs J. E. Field, E. Sommer and U. Soltész for providing me with the beautiful high speed photographs reproduced in the text. I am also greatly indebted to the many experts who have sent me letters, reprints, manuscripts and advice; to Professors R. H. Brown, N. G. W. Cook, K. E. Puttick, H. Rawson, H. E. Rose and K. Schönert; to Drs B. R. Lawn, A. G. Evans, P. F. Messer, R. H. Snow and T. R. Wilshaw; and to many others. Needless to say, none of them must be blamed for my errors or omissions. Finally my best thanks are due to all my colleagues in the department and particularly to Professor J. D. Eshelby and Dr I. C. Howard, for many stimulating discussions and for allowing me to make use of their unpublished work.

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Received 6 June and accepted 11 June 1979.